C3M13

Potential Functions

Before we learn how to evaluate line integrals it is helpful to get some of the terminology straight and learn how different mathematical objects relate to each other. By this time we have established that a function $\vec{F}: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a *vector-valued function* or a *vector field*. For example, $\vec{F}(x,y) = \langle x-y, 2x+y^2 \rangle$ is a simple example of such an object. However, the best and most important example we have is when we take the gradient of a real-valued function of several variables. For example,

$$f(x,y) = e^{2x}\sin 3y \qquad \text{yields the gradient} \qquad \nabla f = \langle 2e^{2x}\sin 3y,\, 3e^{2x}\cos 3y \rangle$$

Definition: A vector field $\vec{F}$ is *conservative* if there is a real-valued function g so the $\nabla g = \vec{F}$. In this case, we say that g is a *potential function* for $\vec{F}$.

Question: Given a vector field, is there a test to determine whether the vector field has a potential function? Or the same question, phrased differently - How do we know when a vector field is conservative?

The answer lies in this theorem. A more expansive version will be stated in a later section.

THEOREM. Suppose $\vec{F}$ is defined on an open simply connected region D. If the component functions of $\vec{F}$ have continuous partial derivatives and the curl of $\vec{F}$ is zero, then $\vec{F}$ is conservative.

(a) (2-dim) If (x_0, y_0) is in D, then we may define a potential function, g, of $\vec{F} = \langle M, N \rangle$ by

$$g(x,y) = \int_{x_0}^{x} M(x,y_0)\,dx \,+\, \int_{y_0}^{y} N(x,y)\,dy$$

(b) If (x_0, y_0, z_0) is in D and $\vec{F} = \langle M, N, P \rangle$, we may define g by

$$g(x,y,z) = \int_{x_0}^{x} M(x,y_0,z_0)\,dx \,+\, \int_{y_0}^{y} N(x,y,z_0)\,dy \,+\, \int_{z_0}^{z} P(x,y,z)\,dz$$

Not only was our question answered, but we are given a method for finding the elusive potential function. There are other methods, but this one is a little more systematic. Before we examine parts (a) and (b) above, let's examine what it means for the *curl of $\vec{F}$ to be zero.* Because we will discuss divergence and curl in a later section, let's just cut to the chase.

(A) If given $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$, then $\vec{F}$ is conservative if, and only if, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(B) If given $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$, then $\vec{F}$ is conservative if, and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

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Part (A) is the two-dimensional statement that the curl is 0, while part (B) is the three-dimensional statement.

Example 1 If $\vec{F}(x,y) = \langle 2x\cos y, -x^2\sin y \rangle$, then $M(x,y) = 2x\cos y$ and $N(x,y) = -x^2\sin y$ so that $\dfrac{\partial N}{\partial x} = -2x\sin y = \dfrac{\partial M}{\partial y}$.

Now we turn to the main objective of this section, specifically, to find the potential function for a conservative vector field. Looking at part (a) of the theorem above we see that $g(x,y)$ is found by evaluating an integral. We must choose a point (x_0, y_0), and it helps to be clever about it, and then integrate to an arbitrary point (x,y). The first integrand is $M(x,y_0)$, which is a pure function of x. Students are reluctant to substitute y_0 for y, but that is exactly what must happen. The second integrand is $N(x,y)$, where we regard x as a constant. In the case of N, the student must be careful to **NOT** substitute either x_0 or y_0 into $N(x,y)$. When the objective is to find $g(x,y)$ so that $\nabla g = \vec{F}$, then $g(x,y) + C$ will also work if g is a solution. The process of finding a potential function is important. After we have learned about line integrals we will do some problems involving conservative vector fields and we will need to find a potential function to make some problems much simpler. Besides, the final exam always has a problem on it requiring the student to find a potential function.

Example 2 Suppose $\vec{F}(x,y) = \left\langle 3x^2\cos(2y) + \dfrac{y}{x},\, -2x^3\sin(2y) + \ln x \right\rangle$

(A) Find the domain D: Because of $\ln x$ we must assume that $x > 0$. Thus, $D = \{(x,y) : x > 0\}$.

(B) Prove that $\vec{F}$ is conservative.

This is done by showing that $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

Note that $\dfrac{\partial N}{\partial x} = -6x^2\sin(2y) + \dfrac{1}{x} = \dfrac{\partial M}{\partial y}$

(C) Find a potential function $g(x,y)$ for $\vec{F}$.

It is important to make a clever choice for (x_0, y_0). Many times the right selection of y_0 eliminates some portion of the first integral that we will use to find g.

Select $(x_0, y_0) = (1, 0)$. Then $M(x,0) = 3x^2\cos(0) + 0/x = 3x^2$ and

$$\begin{aligned}
g(x,y) &= \int_{x_0}^{x} M(x,y_0)\,dx \;+\; \int_{y_0}^{y} N(x,y)\,dy \\
&= \int_1^x 3x^2\,dx + \int_0^y -2x^3\sin(2y) + \ln x\,dy \\
&= x^3\Big|_1^x + x^3\cos(2y) + y\ln x\Big|_0^y \\
&= x^3 - 1 + x^3\cos(2y) + y\ln x - x^3 \\
g(x,y) &= x^3\cos(2y) + y\ln x
\end{aligned}$$

We dropped the -1 from g because the addition of a constant does not affect ∇g.

Example 2, with Maple We will use Maple to solve the problem again. Recall that

$$\vec{F}(x,y) = \left\langle 3x^2\cos(2y) + \frac{y}{x},\, -2x^3\sin(2y) + \ln x \right\rangle$$

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> restart:    with(linalg):
> F:=(x,y)->[3*x^2*cos(2*y)+y/x,-2*x^3*sin(2*y)+ln(x)];
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$$F := (x,y) \to \left[3x^2\cos(2y) + \frac{y}{x},\ -2x^3\sin(2y) + \ln(x)\right]$$

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> F1y:=diff(F(x,y)[1],y);
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$$F1y := -6x^2\sin(2y) + \frac{1}{x}$$

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> F2x:=diff(F(x,y)[2],x);
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$$F2x := -6x^2\sin(2y) + \frac{1}{x}$$

This allows us to conclude that $\vec{F}(x,y)$ is conservative. We choose $(x_0, y_0) = (1, 0)$.

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> x0:=1;  y0:=0
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$$x0 := 1$$
$$y0 := 0$$

Let u be the dummy variable instead of x in the first integral and let v be the dummy variable instead of y in the second integral to establish g.

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> grand1:=F(u,y0)[1];
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$$grand1 := 3u^2$$

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> grand2:=F(x,v)[2];
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$$grand2 := -2x^3\sin(2v) + \ln(x)$$

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> fng:=Int(grand1,u=x0..x)+Int(grand2,v=y0..y);
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$$fng := \int_1^x 3u^2\,du + \int_0^y -2x^3\sin(2v) + \ln(x)\,dv$$

```
> fng:=value(fng);
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$$fng := -x^3 - 1 + 2x^3\cos(y)^2 + \ln(x)\,y$$

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> g:=unapply(fng,(x,y));
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$$g := (x,y) \to -x^3 - 1 + 2x^3\cos(y)^2 + \ln(x)\,y$$

It looks as if Maple has applied the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ so that the function g in the earlier work on Example 2 appears different from that obtained here.

C3M13 problems You may do these by pencil and paper or with Maple. Verify that the given functions are conservative and find a potential function for each.

1. $\vec{F}(x, y) = \langle 3x^2 + 2x + y^2,\, 2xy + y^3 \rangle$
2. $\vec{F}(x, y) = \langle \sin y + \dfrac{y}{x},\, x\cos y + \ln x \rangle$
3. $\vec{F}(x, y) = \langle e^x \cos(y^2) + \dfrac{y}{x^2},\, -2y\, e^x \sin(y^2) - \dfrac{1}{x} \rangle$
4. $\vec{F}(x, y, z) = \langle yz\, e^{xy},\, xz\, e^{xy} - \dfrac{1}{z},\, e^{xy} + \dfrac{y}{z^2} \rangle$